CHARACTERISTIC CYCLES OF LOCAL COHOMOLOGY MODULES OF MONOMIAL IDEALS

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1. Introduction

In his paper [Lyu93] G. Lyubeznik uses the theory of algebraic $\mathcal{D}$-modules to study local cohomology modules. He proves, in particular, that if R is any regular ring containing a field of characteristic zero and $I \subseteq R$ is an ideal, the local cohomology modules $H^i_I(R)$ have the following properties:

i) $H^j_m(H^i_I(R))$is injective, where m is any maximal ideal of R.
ii) $\text{inj.dim}_R(H^i_I(R) \leq \dim_R H^i_I(R)$.
iii) The set of the associated primes of $H^i_I(R)$ is finite.
iv) All the Bass numbers of $H^i_I(R)$ are finite.

By using the Frobenius map, the same results have been obtained by C. Huneke and R.Y. Sharp [HS93] for regular rings containing a field of positive characteristic. By iv), Lyubeznik defines a new set of numerical invariants for any local ring A containing a field, denoted by $\lambda_{p,i}(A)$. Namely, let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension n containing k, and A a local ring which admits a surjective ring homomorphism $\pi : R \longrightarrow A$. Set $I = \operatorname{Ker} \pi$. Then, $\lambda_{p,i}(A)$ is defined as the Bass number $\mu_p(\mathfrak{m}, H^{n-i}_I(R)) = \dim_k \operatorname{Ext}^p_R(k, H^{n-i}_I(R))$. This invariant depends only on A, i and p, but neither on R nor on π. Completion does not change $\lambda_{p,i}(A)$ so if A contains a field but it is not necesarily the quotient of a regular local ring then one can define $\lambda_{p,i}(A) = \lambda_{p,i}(\hat{A})$ where $\hat{A}$ is the completion of A with respect to the maximal ideal. Therefore one can always assume $R = k[[x_1, \ldots, x_n]]$, with $x_1, \ldots, x_n$ independent variables.

In this paper we want to study the local cohomology modules of R supported on a monomial ideal $I \subseteq R$, where $R = k[x_1, \ldots, x_n]$ or $R = k[[x_1, \ldots, x_n]]$ and k is a field of characteristic zero.Since $H^i_I(R) = H^i_{\sqrt{I}}(R)$ we may assume I is reduced. It is well known that reduced monomial ideals have a minimal primary decompositon $I = I_1 \cap \cdots \cap I_m$, where the prime ideals $I_i \subseteq R$ have the form $(x_{i_1}, \ldots, x_{i_h})$, with $x_{i_j} \in \{x_1, \ldots, x_n\}$. Ideals having this form are usually called face ideals. Local cohomology modules supported on a monomial ideal have been studied by G. Lyubeznik in [Lyu84].

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The local cohomology modules $H^r_I(R)$ have a natural structure of finitely generated $D(R,k)$-module, where $D(R,k)$ is the ring of k-linear differential operators of R. In particular, one can define the dimension and the multiplicity of a $D(R,k)$-module. By Bernstein's inequality the dimension of a non-zero finitely generated $D(R,k)$-module is greater or equal to n. The class of $D(R,k)$-modules of dimension n are called holonomic modules and form a category with good properties. The ring R, all the localizations of R by any element of R, as well as the local cohomology modules $H^r_J(R)$ for any ideal J are holonomic. In Section 2 we summarize all these results and recall that to any holonomic module one can attach an invariant, the characterystic cycle, which we study in more detail for R and its localizations by a monomial.

In Section 3 we begin by describing the characteristic cycle of local cohomology modules supported on a face ideal. Then we give the main result, Theorem 3.8, which is a closed formula for the characteristic cycle of any local cohomology module supported on a monomial ideal in terms of the characteristic cycles of the local cohomology modules supported on sums of the face ideals appearing in the minimal primary decomposition.

As a consequence we can decide when a given local cohomology module vanishes and compute the local cohomological dimension $\mathrm{cd}\,(R,I)$ by means of these sums of face ideals, see Corollaries 3.12 and 3.13. We may also give a criterion for the Cohen-Macaulayness of R/I in terms of the minimal primary decompositon, see Corollary 3.11.

Section 4 is dedicated to compute the invariants $\lambda_{p,i}(R/I)$, that we shall call Lyubeznik numbers. Using [Lyu93] we prove that these numbers are exactly the multiplicities of the characteristic cycle of $H^p_{\mathfrak{m}}(H^{n-i}_I(R))$. Then we give a closed formula for these multiplicities, see Theorem 4.4, again in terms of the face ideals appearing in the minimal primary decomposition of I.

We should mention that F. Barkats [Bar95] gives an algorithm to compute a presentation of a local cohomology module supported on a monomial ideal. This algorithm has successfully been performed for ideals in $R = k[x_1,\dots,x_6]$. Then, F. Barkats can describe the characteristic cycle of these modules. Some results on Lyubeznik numbers have also been done by U. Walther for any ideal I. In [Wal96] he describes all possible values of these numbers in the case $\dim(R/I) \leq 2$. The same author, in [Wal97], by using the theory of Gröbner basis on $D(R,k)$-modules, gives algorithms to determine the structure of $H^i_I(R)$ and $H^p_{\mathfrak{m}}(H^i_I(R))$ for arbitrary i,p and find $\lambda_{p,i}(R/I)$. Finally, R. Garcia and C. Sabbah [GS97], express the Lyubeznik numbers of the local ring of a complex isolated singularity in terms of Betti numbers of the associated real link.

2. $\mathcal{D}$-Modules

In this section we fix some notation and recall several results on $\mathcal{D}$-modules. For details see [Bjö79], [Bor87], [Cou95].

Definition 2.1. *Let k be a subring of a commutative noetherian ring R. The ring of differential operators, $D(R,k)$, is the subring of* $\mathrm{Hom}_k(R,R)$ *generated by the k-linear derivations and the multiplications by elements of R.*

By a $D(R,k)$-module we mean a left $D(R,k)$-module.

Set $R = k[x_1,\dots,x_n]$ or $R = k[[x_1,\dots,x_n]]$, where k is a field of characteristic zero and $x_1,\dots,x_n$ are indepentent variables. In both cases $D(R,k)$ is the non commutative R-algebra generated by the partial derivatives $\partial_i = \frac{d}{dx_i}$, with the relations given by:

i) $\partial_i\partial_j = \partial_j\partial_i$.
ii) $\partial_i r + r\partial_i = \frac{dr}{dx_i}$, where $r \in R$.

In the case $R = k[x_1,\dots,x_n]$, $D(R,k)$ is called the n-th Weyl algebra and denoted $A_n(k)$. For simplicity we shall denote in both cases $\mathcal{D} = D(R,k)$. The ring $\mathcal{D}$ is left and right Noetherian .

2.1. Filtrations and graded modules.

Given positive integers $\beta_1,\dots,\beta_n$ set $\partial^\beta = \partial_1^{\beta_1}\cdots\partial_n^{\beta_n}$, $|\beta| = \beta_1+\cdots+\beta_n$. Then:

(1) Every element in $\mathcal{D}$ can be written in a unique way as $\sum q_\beta(x)\partial^\beta$, where $q_\beta(x) \in R$.

(2) The ring $\mathcal{D}$ is equipped with an increasing filtration $\{\Sigma_v\}_{v\geq 0}$, where

$$\Sigma_v = \{Q \in \mathcal{D} : Q = \sum\nolimits_{|\beta|\leq v} q_\beta(x)\partial^\beta\}.$$

For $Q \in \mathcal{D}$, the order of Q is the largest $|\beta|$ for which $q_\beta(x) \neq 0$. So Σ_v is the set of differential operators of order $\leq v$.

(3) The corresponding associated graded ring $gr_\Sigma(\mathcal{D}) = \Sigma_0 \oplus \frac{\Sigma_1}{\Sigma_0} \oplus \cdots$ is isomorphic to the polynomial ring $R[y_1,\dots,y_n]$ where $y_i = \bar{\partial}_i \in \frac{\Sigma_1}{\Sigma_0}$.

(4) A filtration on a $\mathcal{D}$-module M consists of an increasing sequence of R-submodules $\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq M$ satisfying $\forall k, v \geq 0$:

i) $\bigcup \Gamma_k = M$.
ii) $\Sigma_v\Gamma_k \subseteq \Gamma_{v+k}$.

We shall also require each Γ_k to be a finitely generated R-module. Then $gr_\Gamma(M) = \Gamma_0 \oplus \frac{\Gamma_1}{\Gamma_0} \oplus \cdots$ has a natural graded $gr_\Sigma(\mathcal{D})$-module structure.

Definition 2.2. *Let M be a $\mathcal{D}$-module. A filtration Γ on M is good if $gr_\Gamma(M)$ is a finitely generated $gr_\Sigma(\mathcal{D})$-module.*

Proposition 2.3. *M has a good filtration if and only if M is a finitely generated $\mathcal{D}$-module.*

(5) For a finitely generated $\mathcal{D}$-module M and a good filtration Γ on M let $V = \{$maximal ideals of $gr_\Sigma(\mathcal{D})$ contained in the support of $gr_\Gamma(M)\}$. Let $m \in V$ and consider the Hilbert-Samuel function with respect to m:

$$H^0[m, gr_\Gamma(M)](n) = \dim_k\left(\frac{m^n gr_\Gamma(M)}{m^{n+1} gr_\Gamma(M)}\right).$$

For a function $F : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $F(n) = 0$ for $n \ll 0$ one defines:

$$\Delta^{-1}F(n) = \sum_{j=-\infty}^{n} F(j).$$

Then we may consider

$$H^1[m, gr_\Gamma(M)](n) = \Delta^{-1}H^0[m, gr_\Gamma(M)](n)$$

(in this case $F(n) = H^0[m, gr_\Gamma(M)](n) = 0$ for $n < 0$). $H^1[m, gr_\Gamma(M)](n)$ is a polynomial function of degree d_m, so there is a polynomial

$$h^1[m, gr_\Gamma(M)](n) = \frac{e_m}{d_m!} t^{d_m} + \cdots$$

such that $h^1[m, gr_\Gamma(M)](n) = H^1[m, gr_\Gamma(M)](n)$ for $n >> 0$.

The two integers d_m and e_m are called the local dimension and the local multiplicity of $gr_\Gamma(M)$ at the maximal ideal m. We define $d_\Gamma(M) = \max\{d_m | \ m \in V\}$ and $e_\Gamma(M) = \max\{e_m | \ m \in V\}$.

Proposition 2.4. *The integers $d_\Gamma(M)$ and $e_\Gamma(M)$ are the same for all good filtrations on a given finitely generated $\mathcal{D}$-module M.*

We shall put $d(M) = d_\Gamma(M)$ and $e(M) = e_\Gamma(M)$ and call them the dimension and the multiplicity of M.

Theorem 2.5 (Bernstein's inequality)**.** *$d(M) \geq n$ for every non-zero finitely generated $\mathcal{D}$-module M.*

Definition 2.6. *Let M be a finitely generated $\mathcal{D}$-module. One says that M is holonomic if $M = 0$ or $d(M) = n$.*

The class of holonomic modules has many good properties that we shall use in this paper. We list them in the following:

Theorem 2.7. i) *Holonomic modules form a full abelian subcategory of the category of $\mathcal{D}$-modules. In particular if*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is an exact sequence of $\mathcal{D}$-modules, then M_2 is holonomic if and only if M_1 and M_3 are both holonomic.

ii) *If M is holonomic then M has finite length as a $\mathcal{D}$-module.*

iii) *R with its natural structure of $\mathcal{D}$-module is holonomic.*

iv) *If M is holonomic and $f \in R$, then M_f is holonomic.*

v) *If M is holonomic and $I = (f_1, \dots, f_r) \subset R$ then, from the Čech complex:*
$$0 \longrightarrow M \longrightarrow \oplus M_{f_i} \longrightarrow \oplus M_{f_i f_j} \longrightarrow \cdots$$
one concludes by i) and iv) that $H^i_I(M)$ is holonomic for all i.

Remark 2.8. The injective hull of the residue field k, $E_R(k)$ is holonomic since, in this case, it is isomorphic to $H^n_{\mathfrak{m}}(R)$.

Lyubeznik [Lyu93, Proposition 2.3] obtains a presentation of $E_R(k)$ as $\mathcal{D}$-module in the following case:

Proposition 2.9. *Let $R = k[[x_1, ..., x_n]]$ and $\mathfrak{m}$ be the maximal ideal of R. Then as an R-module, $\mathcal{D}/\mathcal{D}\mathfrak{m}$ is isomorphic to $E_R(k)$, the injective hull of the residue field of R in the category of R-modules.*

2.2. **Characteristic Cycle.** Troughout the rest of this section we shall consider the ring $R = \mathbb{C}[x_1, \dots, x_n]$ of polynomials over the field of complex numbers. In this case $gr_\Sigma(\mathcal{D}) = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$. Let M be a finitely generated $\mathcal{D}$-module and Γ a good filtration on M. Consider the ideal in $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$:
$$J(M) = \sqrt{Ann_{gr_\Sigma(\mathcal{D})}(gr_\Gamma(M))}.$$
$J(M)$ is said to be the characteristic ideal of M. One can prove that $J(M)$ is independent of the good filtration on M.

Definition 2.10. *The characteristic variety of M is the closed algebraic set:*
$$C(M) = V(J(M)) \subseteq \operatorname{Spec}(gr_\Sigma(\mathcal{D})).$$

Since $J(M)$ is a homogeneous ideal in the variables $y_1, \dots, y_n$, the characteristic variety has the following property: If $(x, y) \in C(M)$ then $(x, \lambda y) \in C(M)$ for any $\lambda \in \mathbb{C}$. One says that $C(M)$ is a conical variety.

Remark 2.11. Let $X = \operatorname{Spec}(R) = \mathbb{C}^n$. Consider the cotangent bundle $T^*X = \operatorname{Spec}(gr_\Sigma(\mathcal{D})) = \mathbb{C}^{2n}$ and the projection $\pi : T^*X \longrightarrow X$. Then the characteristic variety $C(M)$ may be viewed as a conical subvariety of T^*X.

One can associate a multiplicity to each irreducible component of the characteristic variety in the following way:
Consider the minimal primary decomposition of $J(M)$
$$J(M) = \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_s.$$
Then $(gr_\Gamma(M))_{\mathcal{P}_i}$ is a finitely generated module over $gr_\Sigma(\mathcal{D})_{\mathcal{P}_i}$.

Definition 2.12. *The multiplicity of the irreducible component of the characteristic variety $C(M)$ defined by $\mathcal{P}_i$ is the multiplicity of the $gr_\Sigma(\mathcal{D})_{\mathcal{P}_i}$-module $(gr_\Gamma(M))_{\mathcal{P}_i}$.*

Definition 2.13. *The characteristic cycle of M is defined as:*

$$CC(M) = \sum m_i V_i$$

where the sum is taken over all the irreducible components V_i of $C(M)$ and the m_i's are the corresponding multiplicities.

Proposition 2.14. *The characteristic cycle has the following properties:*

i) *If* $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ *is an exact sequence of holonomic $\mathcal{D}$-modules, then $CC(M_2) = CC(M_1) + CC(M_3)$.*
ii) $CC(M) = 0$ *if and only if* $M = 0$.

As we shall use the results in [BMM94] to compute the characteristic cycle of R and its localizations by a monomial we need to describe those characteristic cycles in terms of conormal bundles relative to a subvariety. For this purpose we need some technical results on Lagrangian varieties that can be found in [Cou95], [Kir88].

Definition 2.15. *Let $Y \subseteq X = \operatorname{Spec}(R)$ be a subvariety, and Y^0 the non-singular part of Y. The conormal bundle relative to Y in X, T_Y^*X, is the closure in $T^*X|_Y$ of*

$$\{v \in T^*X; y = \pi(v) \in Y^0 \textit{ and } v \textit{ annihilates } T_yY\}.$$

Proposition 2.16. *Let $V \subseteq T^*X$ be an irreducible Lagrangian conical closed subvariety of T^*X. Then the image $\pi(V)$ of V under the projection is an irreducible subvariety of X and $V = T^*_{\pi(V)}X$.*

Theorem 2.17. *If M is a holonomic $\mathcal{D}$-module then the characteristic variety $C(M)$ of M is a closed conical Lagrangian subvariety of T^*X.*

Corollary 2.18. *If M is a holonomic $\mathcal{D}$-module then every irreducible component V_i of $C(M)$ is of the form $V_i = T^*_{Y_i}X$ where Y_i is an irreducible subvariety of X.*

Thus we can write

$$CC(M) = \sum m_i T^*_{Y_i}X.$$

2.2.1. *The characteristic cycle of R.* Consider the following filtration of R:

$$\Gamma^0 = \Gamma^1 = \cdots = R.$$

We have

$$gr_\Gamma(R) = R = R[y_1, \ldots, y_n]/(y_1, \ldots, y_n)$$

so Γ is a good filtration, and

$$J(M) = Ann_{gr_\Sigma(\mathcal{D})}(gr_\Gamma(R)) = (y_1, \ldots, y_n).$$

It is easy to see that $C(R) = T^*_X X$, so $CC(R) = T^*_X X$.

2.2.2. *The characteristic cycle of R_f.* A holonomic $\mathcal{D}$-module M is called regular if there exists a filtration Γ such that $Ann_{gr_\Sigma(\mathcal{D})}(gr_\Gamma(R))$ is a radical ideal. If

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is an exact sequence of $\mathcal{D}$-modules, then M_2 is regular holonomic if and only if both M_1 and M_3 are regular holonomic.

In Section 2.2.1 we have seen that R is a regular holonomic $\mathcal{D}$-module, and we shall see that R_f and $H^i_I(R)$ are also regular.

Starting from a result of V. Ginsburg [Gin86, Theorem 3.3], J. Briançon, P. Maisonobe and M. Merle [BMM94] give a geometric formula of the characteristic cycle of M_f in terms of the characteristc cycle of M, where M is a regular holonomic $\mathcal{D}$-module and $f \in R$.

Definition 2.19. *Let $Y \subseteq X$ be an irreducible subvariety such that $f|_Y$ is not a constant, and Y^0 the nonsingular part of Y where $f|_Y$ is a submersion (i.e. $Tf(x) : T_xY \longrightarrow T_{f(x)}X$ is an epimorphism).*

The conormal bundle relative to f , $T_{f|_Y} \subseteq T^*X$, is the closure in $T^*X|_Y$ of:

$$\{v \in T^*X; y = \pi(v) \in Y^0 \text{ and } v \text{ annihilates } T_y((f|_Y)^{-1}(f(y)))\}.$$

Theorem 2.20 ([BMM94]). *Let $CC(M) = \sum m_i T^*_{Y_i}X$ be the characteristic cycle of a regular holonomic $\mathcal{D}$-module M.*

Considering the divisor defined by f on $T_{f|_{Y_i}}$ and the irreducible components, $\Gamma_{i,j}$, of this divisor with $m_{i,j}$ the multiplicity of the ideal defined by $\pi(\Gamma_{i,j})$, let:

$$\Gamma_i = \sum m_{i,j}\Gamma_{i,j}.$$

Then the characteristic cycle of M_f is:

$$CC(M_f) = \sum\nolimits_{f(Y_i)\neq 0} m_i(\Gamma_i + T^*_{Y_i}X).$$

We are going to apply this theorem when f is a monomial on the variables $x_1, \dots, x_n$. We first consider the following two cases:

Case 1: $M = R$ and $f = x_1$. We have $CC(R) = T^*_XX$, so in this case $Y = X$ and $f(X) \neq 0$. By definition, Y^0 is the nonsingular part of X where f is a submersion. We must look for the points in X such that the gradient of f is different from zero. Since $\nabla f = (1, 0, \dots, 0)$ and X is nonsingular, we have $Y^0 = X$. Denote by $\mathcal{C}$ the hypersurface defined by $(f)^{-1}(f(x_1, x_2, \dots, x_n))$. Then $T_{f|_X} \subseteq T^*X$ is the closure of:

$$\{v \in T^*X; z = \pi(v) \in X \text{ and } v \text{ annihilates } T_z\mathcal{C}\}.$$

Since $T_z\mathcal{C} =< (0, -1, \dots, 0), \dots, (0, 0, \dots, -1) >$ we have:

$$T_{f|_X} = \{(x_1, x_2, \dots, x_n, \alpha_1, \alpha_2, \dots, \alpha_n) \in T^*X; \alpha_2 = \dots = \alpha_n = 0\}.$$

So the divisor defined by f on $T_{f|_X}$ is:

$$\Gamma = \{(x_1, x_2, \dots, x_n, \alpha_1, \alpha_2, \dots, \alpha_n) \in T^*X; \alpha_2 = \cdots = \alpha_n = 0, x_1 = 0\}.$$

Note that it has only a component with multiplicity 1, and it is easy to prove that $\Gamma = T^*_{\{x_1=0\}}X$. So

$$CC(R_{x_1}) = T^*_X X + T^*_{\{x_1=0\}}X.$$

Case 2: M is a regular holonomic $\mathcal{D}$-module such that $CC(M) = T^*_{\{x_1=0\}}X$ and $f = x_2$. In this case $Y = Y^0 = \{x_1 = 0\}$ and $f(Y) \neq 0$. Now $\mathcal{C}$ is the hypersurface defined by $(f|_{\{x_1=0\}})^{-1}(f(x_1, x_2, \dots, x_n))$ so:

$$T_{f|_Y} = \{(x_1, x_2, \dots, x_n, \alpha_1, \alpha_2, \dots, \alpha_n) \in T^*X; \alpha_3 = \cdots = \alpha_n = 0, x_1 = 0\}.$$

The divisor defined by f on $T_{f|_Y}$ is:
$\Gamma = \{(x_1, x_2, \dots, x_n, \alpha_1, \alpha_2, \dots, \alpha_n) \in T^*X; \alpha_3 = \cdots = \alpha_n = 0, x_1 = x_2 = 0\} = T^*_{\{x_1=x_2=0\}}X$. So $CC(M_{x_2}) = T^*_{\{x_1=0\}}X + T^*_{\{x_1=x_2=0\}}X$.

By using this two cases we can find the characteristic cycle of R_f for any monomial f. Let $R = \mathbb{C}[x_1, \dots, x_n]$ and $f = x_1 \cdots x_s$, $s \leq n$. Then:

$$\begin{aligned} CC(R_f) = T^*_X X + T^*_{\{x_1=0\}}X + \cdots + T^*_{\{x_s=0\}}X + T^*_{\{x_1=x_2=0\}}X + \\ + \cdots + T^*_{\{x_{s-1}=x_s=0\}}X + \cdots + T^*_{\{x_1=\cdots=x_s=0\}}X. \end{aligned}$$

3. The Main Result

Let $R = k[x_1, \dots, x_n]$ or $R = k[[x_1, \dots, x_n]]$, where k is a field of characteristic zero. In this section we want to describe the characteristic cycle of the local cohomology modules $H^i_I(R)$ for a given ideal $I \subseteq R$ generated by monomials.

3.1. Cohen-Macaulay case.

We begin with the following result, that is a consequence of [Lyu84, Theorem 1] (see also [Bar95, Theorem 5.4.2.2]). In Corollary 3.11 we will reformulate it in terms of the minimal primary decomposition of a monomial ideal.

Recall that $\mathrm{cd}\,(R, I) := \sup\{i; H^i_I(R) \neq 0\}$ is the cohomological dimension of R with respect to I.

Proposition 3.1. *Let $R = k[x_1, \dots, x_n]$ or $R = k[[x_1, \dots, x_n]]$, where k is a field of characteristic zero. For any ideal $I \subseteq R$ generated by monomials, the following are equivalent:*

i) *R/I is Cohen-Macaulay.*
ii) *$H^r_I(R) = 0$ for any $r \neq \mathrm{ht}\, I$.*

Proof. By completion we only have to consider the case $R = k[[x_1, \dots, x_n]]$. We have $\inf\{i; H^i_I(R) \neq 0\} = \mathrm{grade}\, I = \mathrm{ht}\, I$. On the other hand by [Lyu84, Theorem 1 (iv)] we have $\mathrm{cd}\,(R, I) = n - \mathrm{depth}_R(R/I) = \mathrm{ht}\, I$.

□

This result enables us to compute the characteristic cycle $CC(H^h_I(R))$, when I is a monomial ideal of height h such that R/I is Cohen-Macaulay.

Proposition 3.2. *Let $I \in R$ be an ideal generated by monomials $f_1, \ldots, f_r$ and $h = \operatorname{ht} I$. Consider the Čech complex:*

$$0 \longrightarrow R \xrightarrow{d_0} \oplus R_{f_i} \xrightarrow{d_1} \oplus R_{f_i f_k} \xrightarrow{d_2} \cdots \xrightarrow{d_{r-1}} R_{f_1 \cdots f_r} \longrightarrow 0$$

and denote $R_j = \oplus R_{f_{i_1} \cdots f_{i_j}}$, for $j = 0, \ldots, r$. If R/I is Cohen-Macaulay then:

$$CC(H^h_I(R)) = CC(R_h) - CC(R_{h+1}) + \cdots + (-1)^{r-h} CC(R_r)$$
$$-CC(R_{h-1}) + \cdots + (-1)^h CC(R_0).$$

Proof. By Proposition 3.1 $H^r_I(R) = 0$ for any $r \neq h$. Then we have:

$0 = H^0_I(R) = \operatorname{Ker} d_0$. So $CC(\operatorname{Im} d_0) = CC(R_0)$. Similarly $0 = H^1_I(R) = \operatorname{Ker} d_1 / \operatorname{Im} d_0$ and so $CC(\operatorname{Im} d_1) = CC(R_1) - CC(R_0)$. By repeating this argument we obtain

$$CC(Im\ \ d_{h-1}) = CC(R_{h-1}) - CC(R_{h-2}) \cdots + (-1)^h CC(R_0).$$

On the other hand, if $r > h$, we have:
$0 = H^r_I(R) = \operatorname{Ker} d_r / \operatorname{Im} d_{r-1}$, so $CC(\operatorname{Ker} d_r) = CC(R_r)$.
As before

$$CC(\operatorname{Ker} d_h) = CC(R_h) - CC(R_{h+1}) + \cdots + (-1)^{r-h} CC(R_r).$$

Since $H^h_I(R) = \operatorname{Ker} d_h / \operatorname{Im} d_{h-1}$, we get the formula.

□

If I is a complete intersection we then have:

Corollary 3.3. *Assume I is a complete intersection. Then:*
$CC(H^h_I(R)) = CC(R_h) - CC(R_{h-1}) + \cdots + (-1)^h CC(R_0).$

(Note that $CC(R_i)$ have been computed in Section 2.2.2).

We can use this corollary to compute the characteristic cycle of $H^h_I(R)$, where $I = (x_{i_1}, \ldots, x_{i_h})$ is a face ideal. Namely, we have:

$$CC(H^h_I(R)) = T^*_{\{x_{i_1} = \cdots = x_{i_h} = 0\}} X.$$

Remark 3.4. We can extend this result to the case $R = k[x_1, \ldots, x_n]$ or $R = k[[x_1, \ldots, x_n]]$, where k is a field of characteristic zero in the following way:

Case 1: For $L = \mathbb{Q}, \mathbb{C}, k$ denote $R_L = L[x_1, \ldots, x_n]$, and for a face ideal $I \subseteq R_L$ let $M_L = H^h_I(R_L)$. We can consider every face ideal as $I \subseteq R_{\mathbb{Q}}$ and consider the $A_n(\mathbb{Q})$-module $M_{\mathbb{Q}} = H^h_I(R_{\mathbb{Q}})$. By flat base change we have:

$$M_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = H^h_I(R_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C} = H^h_I(R_{\mathbb{C}}) = M_{\mathbb{C}},$$

$$M_{\mathbb{Q}} \otimes_{\mathbb{Q}} k = H^h_I(R_{\mathbb{Q}}) \otimes_{\mathbb{Q}} k = H^h_I(R_k) = M_k.$$

So we get the characteristic ideals $J(M_{\mathbb{C}})$ and $J(M_k)$ as extensions of the characteristic ideal $J(M_{\mathbb{Q}})$ respectively to $\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]$ and $k[x_1,\dots,x_n,y_1,\dots,y_n]$. Then:

$$CC(H_I^h(R_k)) = T^*_{\{x_{i_1}=\cdots=x_{i_h}=0\}}X,$$

where $X = \operatorname{Spec} k[x_1,\dots,x_n]$.

Case 2: Let $\hat{R} = k[[x_1,\dots,x_n]]$. For a face ideal $I \subseteq R = k[x_1,\dots,x_n]$ consider a good filtration $\{\Gamma_v\}$ on $M = H_I^h(R)$. Then $\{\hat{\Gamma_v}\} = \{\Gamma_v \otimes_R \hat{R}\}$ is a good filtration on $\hat{M} = M \otimes_R \hat{R}$, and $gr_{\hat{\Gamma}}\hat{M} = gr_{\Gamma}M \otimes_R \hat{R}$.

By flat base change $\hat{M} = H_I^h(\hat{R})$ and so extending the characteristic ideal of M to $k[[x_1,\dots,x_n]][y_1,\dots,y_n]$ we see that

$$CC(H_I^h(\hat{R})) = T^*_{\{x_{i_1}=\cdots=x_{i_h}=0\}}X,$$

where $X = \operatorname{Spec} k[[x_1,\dots,x_n]]$.

3.2. **General Case.**

3.2.1. *Sum of Face Ideals.* Let $I \subseteq R$ be an ideal generated by squarefree monomials. Consider the minimal primary decompositon $I = I_1 \cap \cdots \cap I_m$.

The ideals I_i are face ideals of the form $(x_{i_1},\dots,x_{i_h})$. These are complete intersection ideals of height h. The sum of face ideals $I_{i_1} + \cdots + I_{i_s}$ is again a face ideal, and $\operatorname{ht}(I_{i_1} + \cdots + I_{i_s}) \leq \operatorname{ht} I_{i_1} + \cdots + \operatorname{ht} I_{i_s}$. From now on we shall denote $h_{i_1\dots i_s} := \operatorname{ht}(I_{i_1} + \cdots + I_{i_s})$.

We are going to describe some sets of sums of the face ideals in the minimal primary decomposition of I They appear in a natural way when we make an iterated use of the Mayer-Vietoris sequence.

Let $I = I_1 \cap \cdots \cap I_m$. Then we define $\omega = \{\omega_1, \omega_2, \dots, \omega_m\}$ where:

$\omega_1 = \{I_1, \dots, I_m\}$,
$\omega_2 = \{I_1 + I_2, \dots, I_{m-1} + I_m\}$,
$\vdots$
$\omega_m = \{I_1 + I_2 + \cdots + I_m\}$.

Similarly, we define the set of sums of face ideals with I_m as a summand, $\omega^m = \{\omega_1^m, \omega_2^m, \dots, \omega_m^m\}$, where:

$\omega_1^m = \{I_m\}$,
$\omega_2^m = \{I_1 + I_m, \dots, I_{m-1} + I_m\}$,
$\vdots$
$\omega_m^m = \{I_1 + I_2 + \cdots + I_m\}$.

Definition 3.5. *We say that $I_{i_1} + \cdots + I_{i_j} \in \omega_j$ and $I_{i_1} + \cdots + I_{i_j} + I_{i_{j+1}} \in \omega_{j+1}$ are paired if $h_{i_1\dots i_j} = h_{i_1\dots i_{j+1}}$. Equivalently, if $I_{i_1} + \cdots + I_{i_j} = I_{i_1} + \cdots + I_{i_j} + I_{i_{j+1}}$, i.e., a generator of $I_{i_{j+1}}$ is also a generator of I_{i_k} for some $k = 1, \dots, j$.*

Definition 3.6. *We say that* $I_{i_1}+\cdots+I_{i_j}\in\omega_j$ *and* $I_{i_1}+\cdots+I_{i_j}+I_{i_{j+1}}\in\omega_{j+1}$ *are almost paired if* $h_{i_1\dots i_j}+1=h_{i_1\dots i_{j+1}}$. *Equivalently, if there is a generator of* $I_{i_{j+1}}$ *which is not a generator of* I_{i_k} *for all* $k=1,\dots,j$.

The formula we shall obtain in Theorem 3.8 for the characteristic cycle of a local cohomology module supported on a monomial ideal will be given in terms of non paired sums of the face ideals in the minimal primary decomposition. The following algorithm allows to obtain these sums.

Algorithm:

Input : $\omega=$ set of sums of face ideals

Output : $\Omega=$ set of non paired sums of face ideals

When we compare a set of sums of face ideals S with a set of sums of face ideals S' we mean the following: If a sum in S is paired with a sum in S' we omit this pair of sums, so repeating this process no sum of face ideals of S is paired to a sum of face ideals of S' at the end. The algorithm is then as follows:

Step 1.

(1.1) Compare I_1+I_2 with $\{I_1,I_2\}$. Save the result in S_0^1.

(1.2) For $k=1,\dots,m-2$; $i_1=3,\dots,m-(k-1)$; $i_2=i_1+1,\dots,m-(k-2)$; $\dots$; $i_k=i_{k-1}+1,\dots,m$; compare $I_1+I_2+I_{i_1}+\cdots+I_{i_k}$ with $\{I_1+I_{i_1}+\cdots+I_{i_k},I_2+I_{i_1}+\cdots+I_{i_k}\}$. Save the result in $S^1_{i_1\dots i_k}$.

For $j=3,\dots,m$;

Step j-1.

(j-1.1) Compare S_j^{j-2} with $\{S_0^{j-2},I_j\}$. Save the result in S_0^{j-1}.

(j-1.2) For $k=1,\dots,m-j$; $i_1=j+1,\dots,m-(k-1)$; $i_2=i_1+1,\dots,m-(k-2)$; $\dots$; $i_k=i_{k-1}+1,\dots,m$; compare $S^{j-2}_{ji_1\dots i_k}$ with $\{S^{j-2}_{i_1\dots i_k},I_j+I_{i_1}+\cdots+I_{i_k}\}$. Save the result in $S^{j-1}_{i_1\dots i_k}$.

The last set of non paired ideals that we obtain by using this algorithm is S_0^{m-1}. Collecting the sums of face ideals of S_0^{m-1} with j summands in Ω_j we get $\Omega=\{\Omega_1,\Omega_2,\dots,\Omega_m\}$, in such a way that no sum in Ω_j is paired with a sum in Ω_{j+1}

We can use a similar algorithm to get the sets of non paired sums of face ideals with the face ideal I_m as a summand, $\Omega^m=\{\Omega_1^m,\Omega_2^m,\dots,\Omega_m^m\}$,

Finally we define the sets of non paired sums of face ideals with a given height $\Omega_{j,r}:=\{I_{i_1}+\cdots+I_{i_j}\in\Omega_j : h_{i_1\dots i_j}=r+(j-1)\}$. The formula we shall give in Theorem 3.8 will be expressed in terms of these sets of non paired sums of face ideals.

Remark 3.7. One can see that the sets $\Omega_{j,r}$ are independent of the sums of face ideals chosen to compare in each step.

We can use a similar algorithm to get the set of non paired sums of face ideals with the face ideal I_m as a summand $\Omega^m = \{\Omega_1^m, \Omega_2^m, \dots, \Omega_m^m\}$, and with a given height $\Omega_{j,r}^m$ $j = 1, \dots, m$.

Example:

Let $R = k[x,y,z,t]$ or $R = k[[x,y,z,t]]$ and $\mathfrak{m} = (x,y,z,t)$. Let $I = (xyz, xyt, xzt)$. Its minimal primary decomposition is $I = (x) \cap (y,z) \cap (z,t) \cap (y,t)$ and the set ω is:

$$\omega_1 = \left\{ \begin{array}{l} I_1 = (x) \\ I_2 = (y,z) \\ I_3 = (z,t) \\ I_4 = (y,t) \end{array} \right\}, \omega_2 = \left\{ \begin{array}{l} I_1 + I_2 = (x,y,z) \\ I_1 + I_3 = (x,z,t) \\ I_2 + I_3 = (y,z,t) \\ I_1 + I_4 = (x,y,t) \\ I_2 + I_4 = (y,z,t) \\ I_3 + I_4 = (y,z,t) \end{array} \right\}, \omega_3 = \left\{ \begin{array}{l} I_1 + I_2 + I_3 = \mathfrak{m} \\ I_1 + I_2 + I_4 = \mathfrak{m} \\ I_1 + I_3 + I_4 = \mathfrak{m} \\ I_2 + I_3 + I_4 = (y,z,t) \end{array} \right\},$$

$$\omega_4 = \left\{ \; I_1 + I_2 + I_3 + I_4 = \mathfrak{m} \; \right\}.$$

The algorithm in this case is as follows:

Step 1.

(1.1) $S_0^1 = \{I_1, I_2, I_1 + I_2\}$.

(1.2) $S_3^1 = \{I_1 + I_3, I_2 + I_3, I_1 + I_2 + I_3\}$,
$S_4^1 = \{I_1 + I_4, I_2 + I_4, I_1 + I_2 + I_4\}$,
$S_{34}^1 = \{I_2 + I_3 + I_4\}$.

Step 2.

(2.1) $S_0^2 = \{I_1, I_2, I_3, I_1 + I_2, I_1 + I_3, I_2 + I_3, I_1 + I_2 + I_3\}$.

(2.2) $S_4^2 = \{I_1 + I_4, I_3 + I_4, I_1 + I_2 + I_4\}$.

Step 3.

(3.1) $S_0^3 = \{I_1, I_2, I_3, I_4, I_1 + I_2, I_1 + I_3, I_2 + I_3, I_1 + I_4, I_3 + I_4, I_1 + I_2 + I_3, I_1 + I_2 + I_4\}$.

So we have:
$\Omega_1 = \{I_1, I_2, I_3, I_4\}$,
$\Omega_2 = \{I_1 + I_2, I_1 + I_3, I_2 + I_3, I_1 + I_4, I_3 + I_4\}$,
$\Omega_3 = \{I_1 + I_2 + I_3, I_1 + I_2 + I_4\}$,
$\Omega_4 = \{\emptyset\}$.

Collecting the sums of face ideals by the heights we obtain:
$\Omega_{1,1} = \{I_1\}$,
$\Omega_{1,2} = \{I_2, I_3, I_4\}$,
$\Omega_{2,2} = \{I_1 + I_2, I_1 + I_3, I_2 + I_3, I_1 + I_4, I_3 + I_4\}$,
$\Omega_{3,2} = \{I_1 + I_2 + I_3, I_1 + I_2 + I_4\}$.

Now we are ready to formulate our main result.

Theorem 3.8. *Let $I \subseteq R$ be an ideal generated by squarefree monomials and let $I = I_1 \cap \cdots \cap I_m$ be the minimal primary decompositon. Then :*

$$CC(H^r_I(R)) = \sum_{I_i \in \Omega_{1,r}} CC(H^r_{I_i}(R)) + \sum_{I_i+I_j \in \Omega_{2,r}} CC(H^{r+1}_{I_i+I_j}(R)) + \cdots +$$
$$+ \sum_{I_1+\cdots+I_m \in \Omega_{m,r}} CC(H^{r+(m-1)}_{I_1+\cdots+I_m}(R)).$$

Proof. We shall proceed by induction on m, the number of ideals in the minimal primary decomposition, being the case $m = 1$ trivial. To do it we shall split a Mayer-Vietoris sequence of the type:

$$\cdots \longrightarrow H^r_{U+V}(R) \longrightarrow H^r_U(R) \oplus H^r_V(R) \longrightarrow H^r_{U\cap V}(R) \longrightarrow H^{r+1}_{U+V}(R) \longrightarrow \cdots$$

into short exact sequences of kernels and cokernels:

$$0 \longrightarrow B_r \longrightarrow H^r_U(R) \oplus H^r_V(R) \longrightarrow C_r \longrightarrow 0,$$

$$0 \longrightarrow C_r \longrightarrow H^r_{U\cap V}(R) \longrightarrow A_{r+1} \longrightarrow 0,$$

$$0 \longrightarrow A_{r+1} \longrightarrow H^{r+1}_{U+V}(R) \longrightarrow B_{r+1} \longrightarrow 0,$$

so:

$$CC(H^r_{U\cap V}(R)) = CC(C_r) + CC(A_{r+1}).$$

Assume we have proved the formula for ideals with less terms than m in the minimal primary decomposition. Now consider:
$U = I_1 \cap \cdots \cap I_{m-1}$,
$V = I_m$,
$U \cap V = I = I_1 \cap \cdots \cap I_m$,
$U + V = I_1 \cap \cdots \cap I_{m-1} + I_m$.
By induction we have determined $CC(H^r_U(R))$ and $CC(H^r_V(R))$. Now we shall describe $CC(H^r_{U+V}(R))$.

Lemma 3.9. *Let $I \subseteq R$ be generated by squarefree monomials and $I = I_1 \cap \cdots \cap I_m$ be the minimal primary decompositon. Then :*

$$CC(H^r_{I_1\cap\cdots\cap I_{m-1}+I_m}(R)) = \sum_{I_i+I_m \in \Omega^m_{2,r}} CC(H^r_{I_i+I_m}(R)) + \cdots +$$
$$+ \sum_{I_1+\cdots+I_m \in \Omega^m_{m,r}} CC(H^{r+(m-2)}_{I_1+\cdots+I_m}(R)).$$

Remark 3.10. The formulas in Theorem 3.8 and Lemma 3.9 show that if $\alpha \in CC(H^r_I(R))$ then $\alpha \notin CC(H^j_I(R))$ for $j \neq r$.

Proof. We use induction on the number of components in the minimal primary decompositon. We shall consider a Mayer-Vietoris sequence with

$U = I_1 \cap \cdots \cap I_{m-2} + I_m$,
$V = I_{m-1} + I_m$,
$U \cap V = I_1 \cap \cdots \cap I_{m-1} + I_m$,
$U + V = I_1 \cap \cdots \cap I_{m-2} + (I_{m-1} + I_m)$.

We have by induction

$$CC(H^r_{I_1\cap\cdots\cap I_{m-2}+I_m}(R)) = \sum_{I_i+I_m\in\Omega^m_{2,r}} CC(H^r_{I_i+I_m}(R)) + \cdots +$$
$$+ \sum_{I_1+\cdots+I_m\in\Omega^m_{m,r}} CC(H^{r+(m-3)}_{I_1+\cdots+I_{m-2}+I_m}(R)),$$

$CC(H^r_{I_{m-1}+I_m}(R)) = CC(H^r_{I_{m-1}+I_m}(R))$, and

$$CC(H^r_{I_1\cap\cdots\cap I_{m-2}+(I_{m-1}+I_m)}(R)) =$$
$$= \sum_{I_i+I_{m-1}+I_m\in\Omega^{(m-1)+m}_{2,r}} CC(H^r_{I_i+I_{m-1}+I_m}(R)) + \cdots +$$
$$+ \sum_{I_1+\cdots+I_m\in\Omega^{(m-1)+m}_{m,r}} CC(H^{r+(m-3)}_{I_1+\cdots+I_m}(R)).$$

Note that we use the sets $\Omega^{(m-1)+m}_{j,r}$ of non paired sums of face ideals with the face ideal $I_{m-1}+I_m$ as a summand to describe $CC(H^r_{U+V}(R))$. To get the desired formula we only need to describe $CC(B_r)$, so it is enough to prove the following

Claim:

$$CC(B_r) = \sum T^*_{\{x_{i_1}=\cdots=x_{i_j}=0\}}X,$$

where the sum is taken over the cycles

$$T^*_{\{x_{i_1}=\cdots=x_{i_j}=0\}}X \in CC(H^r_U(R) \oplus H^r_V(R)) \cap CC(H^r_{U+V}(R)).$$

The inclusion $\subseteq$ is obvious. To prove the other one let $\alpha = T^*_{\{x_{i_1}=\cdots=x_{i_j}=0\}}X \in CC(H^r_U(R) \oplus H^r_V(R)) \cap CC(H^r_{U+V}(R))$ and suppose $\alpha \notin CC(B_r)$. We must consider the sum of face ideals in the minimal primary decomposition that we need to express α as the characteristic cycle of a local cohomology module supported on this sum.

Case 1: $\alpha = CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_m}(R)) = CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_{m-1}+I_m}(R))$, where $s < m-2$.

(1.1) If there exists $l \in \{1, \ldots, m-2\} \smallsetminus \{i_1, \ldots, i_s\}$ such that $\alpha \neq CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_l+I_m}(R))$, then I_l is not involved in expressing α and one can consider the following Mayer-Vietoris sequence

$U = I_1 \cap \cdots \cap \hat{I}_l \cap \cdots \cap I_{m-2} + I_m$,
$V = I_{m-1} + I_m$,
$U \cap V = I_1 \cap \cdots \cap \hat{I}_l \cap \cdots \cap I_{m-1} + I_m$,
$U + V = I_1 \cap \cdots \cap \hat{I}_l \cap \cdots \cap I_{m-2} + (I_{m-1} + I_m)$.

Note that in this sequence $\alpha \in CC(H^r_U(R) \oplus H^r_V(R)) \cap CC(H^r_{U+V}(R))$ and $\alpha \notin CC(B_r)$.

By induction and using Remark 3.10 there is a contradiction since:

$$\alpha \in CC(H^r_{U+V}(R)) \text{ and } \alpha \notin CC(B_r) \Longrightarrow \alpha \in CC(H^{r-1}_{U\cap V}(R)),$$

$$\alpha \in CC(H^r_U(R) \oplus H^r_V(R)) \text{ and } \alpha \notin B_r \Longrightarrow \alpha \in CC(H^r_{U\cap V}(R)).$$

(1.2) If for any $l \in \{1, \ldots, m-2\} \setminus \{i_1, \ldots, i_s\}$ we have $\alpha = CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_l+I_m}(R))$ then:

$$\alpha = CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_l+I_m}(R)) \notin CC(H^r_{I_1\cap\cdots\cap I_{m-2}+I_m}(R) \oplus H^r_{I_{m-1}+I_m}(R)),$$

and so there is a previous induction step in computing

$$CC(H^r_{I_1\cap\cdots\cap I_{m-2}+I_m}(R) \oplus H^r_{I_{m-1}+I_m}(R))$$

such that $\alpha = CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_l+I_m}(R)) \in CC(B_r)$. Recall that $CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_m}(R)) \notin CC(B_r)$ for any step, so there exists J, sum of face ideals with $I_l + I_m$ as a summand, such that

$$\alpha = CC(H^{r+(s-1)}_J(R)) \notin CC(H^r_{I_1\cap\cdots\cap I_{m-2}+I_m}(R) \oplus H^r_{I_{m-1}+I_m}(R)).$$

We also have

$$\alpha = CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_l+I_{m-1}+I_m}(R)) \notin CC(H^r_{I_1\cap\cdots\cap I_{m-2}+(I_{m-1}+I_m)}(R)),$$

so $CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_l+I_{m-1}+I_m}(R)) \in CC(B_r)$ in the corresponding induction step and

$$\alpha = CC(H^{r+(s-1)}_{J+I_{m-1}}(R)) \notin CC(H^r_{I_1\cap\cdots\cap I_{m-2}+(I_{m-1}+I_m)}(R)).$$

Then, when we use the Mayer-Vietoris sequence with
$U = I_1 \cap \cdots \cap \hat{I}_l \cap \cdots \cap I_{m-2} + I_m$,
$V = I_{m-1} + I_m$,
$U \cap V = I_1 \cap \cdots \cap \hat{I}_l \cap \cdots \cap I_{m-1} + I_m$,
$U + V = I_1 \cap \cdots \cap \hat{I}_l \cap \cdots \cap I_{m-2} + (I_{m-1} + I_m)$,
we have

$$\alpha = CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_m}(R)) \in CC(H^r_U(R) \oplus H^r_V(R)),$$

$$\alpha = CC(H^{r+(s-1)}_{I_{i_1}+\cdots+I_{i_s}+I_{m-1}+I_m}(R)) \in CC(H^r_{U+V}(R))$$

and $\alpha \notin CC(B_r)$, and so we get a contradiction.

Case 2: $\alpha = CC(H^{r+(m-3)}_{I_1+\cdots+I_{m-2}+I_m}(R)) = CC(H^{r+(m-3)}_{I_1+\cdots+I_{m-2}+I_{m-1}+I_m}(R))$.

In this case we consider the Mayer-Vietoris sequence with

$U=(I_1+\cdots+I_{m-2})+I_m,$
$V=I_{m-1}+I_m,$
$U\cap V=(I_1+\cdots+I_{m-2})\cap I_{m-1}+I_m,$
$U+V=(I_1+\cdots+I_{m-2})+(I_{m-1}+I_m).$
we have

$$\alpha=CC(H^{r+(m-3)}_{I_1+\cdots+I_{m-2}+I_m}(R))\in CC(H^r_U(R)\oplus H^r_V(R)),$$

$$\alpha=CC(H^{r+(m-3)}_{I_1+\cdots+I_{m-2}+I_{m-1}+I_m}(R))\in CC(H^r_{U+V}(R))$$

and $\alpha\notin CC(B_r)$, so we get a contradiction and this proves the claim.

□

Now we may continue the proof of Theorem 3.8. Recall that we use induction on the number of components in the minimal primary decompositon and the Mayer-Vietoris sequence:

$$\cdots\longrightarrow H^r_{I_1\cap\cdots\cap I_{m-1}}(R)\oplus H^r_{I_m}(R)\longrightarrow H^r_I(R)\longrightarrow H^{r+1}_{I_1\cap\cdots\cap I_{m-1}+I_m}(R)\longrightarrow\cdots.$$

By Lemma 3.9 and induction we have:

$$CC(H^r_{I_1\cap\cdots\cap I_{m-1}}(R))=\sum_{I_i\in\Omega_{1,r}}CC(H^r_{I_i}(R))+\cdots+$$
$$+\sum_{I_1+\cdots+I_{m-1}\in\Omega_{m-1,r}}CC(H^{r+(m-2)}_{I_1+\cdots+I_{m-1}}(R)),$$

$$CC(H^r_{I_1\cap\cdots\cap I_{m-1}+I_m}(R))=\sum_{I_i+I_m\in\Omega^m_{2,r}}CC(H^r_{I_i+I_m}(R))+\cdots+$$
$$+\sum_{I_1+\cdots+I_m\in\Omega^m_{m,r}}CC(H^{r+(m-2)}_{I_1+\cdots+I_m}(R)).$$

To finish the proof of Theorem 3.8 we must only describe $CC(B_r)$.
Claim:

$$CC(B_r)=\sum T^*_{\{x_{i_1}=\cdots=x_{i_j}=0\}}X,$$

where the sum is taken over the cycles $T^*_{\{x_{i_1}=\cdots=x_{i_j}=0\}}X\in CC(H^r_{I_1\cap\cdots\cap I_{m-1}}(R)\oplus H^r_{I_{m-1}+I_m}(R))\cap CC(H^r_{I_1\cap\cdots\cap I_{m-1}+I_m}(R))$.

The proof of this claim is as in Lemma 3.9. □

Example: Let $R=k[x,y,z,t]$ or $R=k[[x,y,z,t]]$. For the ideal $I=(xyz,xyt,xzt)\subseteq R$ we have already computed the sets $\Omega_{j,r}$ of non paired sums of face ideals in the minimal primary decomposition of I with a given height. Therefore we can describe the characteristic cycle of the local cohomology modules supported on I. Namely,

$$CC(H^1_I(R))=CC(H^1_{I_1}(R)).$$

$$CC(H^2_I(R))=CC(H^2_{I_2}(R))+CC(H^2_{I_3}(R))+CC(H^2_{I_4}(R))+$$
$$+CC(H^3_{I_1+I_2}(R))+CC(H^3_{I_1+I_3}(R))+CC(H^3_{I_2+I_3}(R))+$$
$$+CC(H^3_{I_1+I_4}(R))+CC(H^3_{I_3+I_4}(R))+CC(H^4_{I_1+I_2+I_3}(R))+$$
$$+CC(H^4_{I_1+I_2+I_4}(R)).$$

In terms of conormal bundles relative to a subvariety we have:

$CC(H_I^1(R)) = T^*_{\{x=0\}}X.$

$CC(H_I^2(R)) = T^*_{\{y=z=0\}}X + T^*_{\{z=t=0\}}X + T^*_{\{y=t=0\}}X+$
$+T^*_{\{x=y=z=0\}}X + T^*_{\{x=z=t=0\}}X + T^*_{\{x=y=t=0\}}X + 2T^*_{\{y=z=t=0\}}X+$
$+2T^*_{\{x=y=z=t=0\}}X.$

Note that there are two local cohomology modules different from zero, so R/I is not Cohen-Macaulay by Proposition 3.1.

By using Theorem 3.8 we can also reformulate Proposition 3.1 in terms of the sets of non paired sums of face ideals.

Corollary 3.11. *Let $R = k[x_1, \dots, x_n]$ or $R = k[[x_1, \dots, x_n]]$, where k is a field of characteristic zero. For an ideal $I \subseteq R$ generated by squarefree monomials, the following are equivalent:*

i) *R/I is Cohen-Macaulay.*
ii) *$H_I^r(R) = 0$ for any $r \neq \operatorname{ht} I$.*
iii) *$\Omega_{j,r} = \emptyset$, for any $r \neq \operatorname{ht} I$, $\forall j$.*
iv) *For all $I_{i_1} + \cdots + I_{i_j} \in \Omega_j$ $h_{i_1 \dots i_j} = \operatorname{ht} I + (j-1)$.*

Note that Theorem 3.8 also provides a criterion to decide when $H_I^r(R)$ vanishes. By [Lyu84, Theorem 1,(iii)] this is equivalent to determine when $\operatorname{Ext}_R^r(R/I, R)$ vanishes.

Corollary 3.12. *Let $R = k[x_1, \dots, x_n]$ or $R = k[[x_1, \dots, x_n]]$, where k is a field of characteristic zero. For an ideal $I \subseteq R$ generated by squarefree monomials, the following are equivalent:*

i) *$H_I^r(R) \neq 0$.*
ii) *There exists j such that $\Omega_{j,r} \neq \emptyset$.*
iii) *There exists $I_{i_1} + \cdots + I_{i_j} \in \Omega_j$ such that $h_{i_1 \dots i_j} = r + (j-1)$.*

Corollary 3.13. *The cohomological dimension of R with respect to I is:*
$\operatorname{cd}(R, I) = \mathit{max}\ \{h_{i_1 \dots i_j} - (j-1) : I_{i_1} + \cdots + I_{i_j} \in \Omega_j\}$.

Remark 3.14. By [Lyu84, Theorem 1,(iv)] $\operatorname{cd}(R, I) = \operatorname{proj.dim}_R(R/I) = n - \operatorname{depth}_R(R/I)$.

4. Lyubeznik Numbers

Let A be a quotient of dimension d of the regular local ring $R = k[[x_1, \dots, x_n]]$, with $x_1, \dots, x_n$ independent variables. In this section we want to study the Lyubeznik numbers $\lambda_{p,i}(A)$ introduced in [Lyu93, §4]. In his paper Lyubeznik also gives some properties of these numbers:

i) $\lambda_{p,i}(A) = 0$ if $i > d$.
ii) $\lambda_{p,i}(A) = 0$ if $p > i$.
iii) $\lambda_{d,d}(A) \neq 0$.

Walther [Wal96] defines the type of R/I as the tringular matrix given by $\lambda_{p,i}(A)$.

$$\Lambda(A) = \begin{pmatrix} \lambda_{0,0} & \cdots & \lambda_{0,d} \\ & \ddots & \vdots \\ & & \lambda_{d,d} \end{pmatrix}.$$

Remark 4.1. By [Lyu93, Lemma 1.4]:

$$\lambda_{p,i}(R/I) = \mu_p(\mathfrak{m}, H_I^{n-i}(R)) = \mu_0(\mathfrak{m}, H_{\mathfrak{m}}^p(H_I^{n-i}(R))).$$

4.1. **Lyubeznik numbers and characteristic cycles.** Let $R = k[[x_1, \ldots, x_n]]$ and $I \subseteq R$ an ideal generated by monomials. In this section we shall compute the characteristic cycle of $H_{\mathfrak{m}}^p(H_I^r(R))$. Consider the ring of differential operators $\mathcal{D} = D(R,k)$. By [Lyu93, Theorem 3.4 (a)]

$$H_{\mathfrak{m}}^p(H_I^{n-i}(R)) = (\mathcal{D}/\mathcal{D}\mathfrak{m})^{\lambda_{p,i}(R/I)} = E_R(R/\mathfrak{m})^{\lambda_{p,i}(R/I)}.$$

Consider the filtration $\{\Sigma_v\}$ on $\mathcal{D}$. Then $\{\Sigma_v/\Sigma_v \cap \mathfrak{m}\}$ and $\{\Sigma_v \cap \mathfrak{m}\}$ are good filtrations on $\mathcal{D}/\mathcal{D}\mathfrak{m}$ and $\mathcal{D}\mathfrak{m}$ respectively. We have an exact sequence

$$0 \longrightarrow gr\mathcal{D}\mathfrak{m} \longrightarrow gr\mathcal{D} \longrightarrow gr\mathcal{D}/\mathcal{D}\mathfrak{m} \longrightarrow 0.$$

Thus $gr\mathcal{D}/\mathcal{D}\mathfrak{m} \cong gr\mathcal{D}/gr\mathcal{D}\mathfrak{m}$, and one can see that the characteristic ideal is $J(\mathcal{D}/\mathcal{D}\mathfrak{m}) = \mathfrak{m}$. So $CC(\mathcal{D}/\mathcal{D}\mathfrak{m}) = T^*_{\{x_1=\cdots=x_n=0\}}X$.

Then $CC(H_{\mathfrak{m}}^p(H_I^{n-i}(R))) = \lambda_{p,i} T^*_{\{x_1=\cdots=x_n=0\}}X$.

Remark 4.2. If $I \subseteq R$ is a monomial ideal of height h such that R/I is Cohen-Macaulay then the spectral sequence $E_2^{p,q} = H_{\mathfrak{m}}^p(H_I^q(R)) \Longrightarrow H_{\mathfrak{m}}^{p+q}(R)$ gives:

$$H_{\mathfrak{m}}^p(H_I^h(R)) = \begin{cases} 0 & if \quad p \neq n-h, \\ E_R(R/\mathfrak{m}) & if \quad p = n-h. \end{cases}$$

So the type is:

$$\Lambda(R/I) = \begin{pmatrix} 0 & \cdots & 0 \\ & \ddots & \vdots \\ & & 1 \end{pmatrix}.$$

Later we shall see by using the example with $R = k[[x,y,z,t]]$ and $I = (xyz, xyt, xzt)$ that the converse does not hold.

4.1.1. *Sum Ideals.* The formula we shall obtain in Theorem 4.4 for the characteristic cycle of $H_{\mathfrak{m}}^p(H_I^r(R))$ will be given in terms of non paired and non almost paired sums of the face ideals in the minimal primary decomposition. We give an analogous algorithm to the one of Section 3.2.1 to get these sums.

Algorithm:

Input : $\Omega =$ set of non paired sums of face ideals.

Output : $\Gamma =$ set of non paired and non almost paired sums of face ideals.

The algorithm is as the one of Section 3.2.1 but now, when we compare a set of non paired sums of face ideals S with a set of non paired sums of face ideals S' we mean the following: If a sum in S is almost paired with a sum in S' we omit this pair of sums, so repeating this process no sum of non paired face ideals of S is almost paired to a sum of non paired face ideals of S' at the end.

Collecting by the number of summands the last set of non paired and non almost paired sums of face ideals obtained by using this algorithm we get $\Gamma = \{\Gamma_1, \Gamma_2, \dots, \Gamma_m\}$, in such a way that no sum in Γ_j is almost paired with a sum in Γ_{j+1}.

We also define the set of non paired and non almost paired sums of face ideals with a given height $\Gamma_{j,r} = \{I_{i_1} + \cdots + I_{i_j} \in \Gamma_j : h_{i_1 \dots i_j} = r + (j-1)\}$. We shall denote $h_{j,r} = r + (j-1)$. The formula we shall give in Theorem 4.4 will be expressed in terms of these sets of non paired and non almost paired sums of face ideals.

Remark 4.3. One can see that the numbers of face ideals in the sets $\Gamma_{j,r}$ are independent of the sums of face ideals chosen to compare in each step.

We can use a similar algorithm to get the sets of non paired and non almost paired sums of face ideals with the face ideal I_m as a summand, $\Gamma^m = \{\Gamma^m_1, \Gamma^m_2, \dots, \Gamma^m_m\}$, and with a given height $\Gamma^m_{j,r}$ for $j = 1, \dots, m$.

Example:

Let $R = k[x,y,z,t]$ or $R = k[[x,y,z,t]]$ and $\mathfrak{m} = (x,y,z,t)$. Let $I = (xyz, xyt, xzt)$; its minimal primary decomposition is $I = (x) \cap (y,z) \cap (z,t) \cap (y,t)$ and the set Ω is:

$$\Omega_1 = \left\{ \begin{array}{c} I_1 = (x) \\ I_2 = (y,z) \\ I_3 = (z,t) \\ I_4 = (y,t) \end{array} \right\}, \Omega_2 = \left\{ \begin{array}{c} I_1 + I_2 = (x,y,z) \\ I_1 + I_3 = (x,z,t) \\ I_2 + I_3 = (y,z,t) \\ I_1 + I_4 = (x,y,t) \\ I_3 + I_4 = (y,z,t) \end{array} \right\}, \Omega_3 = \left\{ \begin{array}{c} I_1 + I_2 + I_3 = \mathfrak{m} \\ I_1 + I_2 + I_4 = \mathfrak{m} \end{array} \right\},$$

$$\Omega_4 = \{\ \emptyset\ \}.$$

The algorithm is then as follows:

Step 1.

(1.1) $S^1_0 = \{I_1\}$.

(1.2) $S^1_3 = \{I_1 + I_3\}$,

$S^1_4 = \{\emptyset\}$,

$S^1_{3,4} = \{\emptyset\}$.

Step 2.
(2.1) $S_0^2 = \{I_1\}$.
(2.2) $S_4^2 = \{I_3 + I_4\}$.
Step 3.
(3.1) $S_0^3 = \{I_1\}$.

So we have:
$\Gamma_1 = \Gamma_{1,1} = \{I_1\}$, $\Gamma_2 = \{\emptyset\}$, $\Gamma_3 = \{\emptyset\}$, $\Gamma_4 = \{\emptyset\}$.

Theorem 4.4. *Let $I \subseteq R = k[[x_1, \dots, x_n]]$ be an ideal generated by squarefree monomials and $I = I_1 \cap \dots \cap I_m$ be its minimal primary decompositon. Then :*

$$CC(H^p_{\mathfrak{m}}(H^r_I(R))) = \lambda_{p,n-r} T^*_{\{x_1 = \dots = x_n = 0\}} X,$$

where $\lambda_{p,n-r} = \sharp\Gamma_{j,r}$ such that $h_{j,r} = n - p$.

Proof. The proof is similar to Theorem 3.8 and we shall use the same notation. Consider the Mayer-Vietoris sequence with
$U = I_1 \cap \dots \cap I_{m-1}$,
$V = I_m$,
$U \cap V = I = I_1 \cap \dots \cap I_m$,
$U + V = I_1 \cap \dots \cap I_{m-1} + I_m$.
Then we have:

$$0 \longrightarrow C_r \longrightarrow H^r_{U \cap V}(R) \longrightarrow A_{r+1} \longrightarrow 0.$$

Applying the long exact sequence of local cohomology we get:

$$\cdots \longrightarrow H^p_{\mathfrak{m}}(C_r) \longrightarrow H^p_{\mathfrak{m}}(H^r_{U \cap V}(R)) \longrightarrow H^p_{\mathfrak{m}}(A_{r+1}) \longrightarrow H^{p+1}_{\mathfrak{m}}(C_r) \longrightarrow \cdots .$$

We shall split this sequence into short exact sequences of kernels and cokernels:

$$0 \longrightarrow Z_{p-1} \longrightarrow H^p_{\mathfrak{m}}(C_r) \longrightarrow X_p \longrightarrow 0$$

$$0 \longrightarrow X_p \longrightarrow H^p_{\mathfrak{m}}(H^r_{U \cap V}(R)) \longrightarrow Y_p \longrightarrow 0$$

$$0 \longrightarrow Y_p \longrightarrow H^p_{\mathfrak{m}}(A_{r+1}) \longrightarrow Z_p \longrightarrow 0.$$

So $CC(H^p_{\mathfrak{m}}(H^r_{U \cap V}(R))) = CC(X_p) + CC(Y_p)$.

Assume we have proved the formula for ideals with less terms than m in the minimal primary decomposition. To compute $CC(H^p_{\mathfrak{m}}(A_{r+1}))$ we need to describe $CC(H^p_{\mathfrak{m}}(H^{r+1}_{I_1 \cap \dots \cap I_{m-1} + I_m}(R)))$.

Lemma 4.5. *Let $I \subseteq R = k[[x_1, \dots, x_n]]$ be generated by squarefree monomials and $I = I_1 \cap \dots \cap I_m$ be the minimal primary decompositon. Then:*

$$CC(H^p_{\mathfrak{m}}(H^r_{I_1 \cap \dots \cap I_{m-1} + I_m}(R))) = \lambda_{p,n-r} T^*_{\{x_1 = \dots = x_n = 0\}} X,$$

where $\lambda_{p,n-r} = \sharp\Gamma^m_{j,r}$ such that $h^m_{j,r} = n - p$.

Proof. We shall use induction on m, the number of components in the minimal primary decompositon.

$m=2$.

Let $h = \operatorname{ht}(I_1+I_2)$. Since I_1+I_2 is a complete intersection, by using Remark 4.2 we obtain:

$$CC(H_m^{n-h}(H^h_{I_1+I_2}(R)) = T^*_{\{x_1=\cdots=x_n=0\}}X.$$

$m>2$.

Consider the Mayer-Vietoris sequence with

$U = I_1\cap\cdots\cap I_{m-2}+I_m$,
$V = I_{m-1}+I_m$,
$U\cap V = I_1\cap\cdots\cap I_{m-1}+I_m$,
$U+V = I_1\cap\cdots\cap I_{m-2}+(I_{m-1}+I_m)$.

Recall that $CC(C_r) \subseteq CC(H^r_U(R)) + CC(H^r_V(R))$ and $CC(A_{r+1}) \subseteq CC(H^{r+1}_{U+V}(R))$. Where:

$$CC(H^r_U(R)) = \textstyle\sum_{I_i+I_m\in\Omega^m_{2,r}} CC(H^r_{I_i+I_m}(R)) + \cdots +$$
$$+\textstyle\sum_{I_1+\cdots+I_m\in\Omega^m_{m,r}} CC(H^{r+(m-3)}_{I_1+\cdots+I_{m-2}+I_m}(R)),$$

$CC(H^r_V(R)) = CC(H^r_{I_{m-1}+I_m}(R))$, and

$$CC(H^{r+1}_{U+V}(R)) = \textstyle\sum_{I_i+I_{m-1}+I_m\in\Omega^{(m-1)+m}_{2,r+1}} CC(H^{r+1}_{I_i+I_{m-1}+I_m}(R)) + \cdots +$$
$$+\textstyle\sum_{I_1+\cdots+I_m\in\Omega^{(m-1)+m}_{m,r+1}} CC(H^{r+(m-2)}_{I_1+\cdots+I_m}(R)).$$

Note that we use the sets $\Omega^{(m-1)+m}_{j,r+1}$ of non paired sums of face ideals with the face ideal $I_{m-1}+I_m$ as a summand to describe $CC(H^{r+1}_{U+V}(R))$. To get the desired formula we only need to describe $CC(Z_p)$, so it is enough to prove the following

Claim:

$$CC(Z_p) = \sum T^*_{\{x_1=\cdots=x_n=0\}}X,$$

where the sum is taken over the cycles

$$T^*_{\{x_1=\cdots=x_n=0\}}X = CC(H^p_{\mathfrak{m}}(H^{r+j}_{I_{i_1}+\cdots+I_{i_j}+I_{m-1}+I_m}(R))) \in CC(H^p_{\mathfrak{m}}(A_{r+1})),$$

such that

$$T^*_{\{x_1=\cdots=x_n=0\}}X = CC(H^{p+1}_{\mathfrak{m}}(H^{r+j-1}_{I_{i_1}+\cdots+I_{i_j}+I_m}(R))) \in CC(H^{p+1}_{\mathfrak{m}}(C_r)).$$

Note that if $\alpha\in CC(A_{r+1})$ then $\alpha\notin CC(A_j)$ for $j\neq r+1$, and if $\alpha\in CC(C_r)$ then $\alpha\notin CC(C_j)$ for $j\neq r$. Now the proof follows as Lemma 3.9. $\square$

Now we may continue the proof of Theorem 4.4. Recall that we use induction on m, the number of components in the minimal primary decompositon.

$m = 2$.

Let $I = I_1 \cap I_2$. By using the Mayer-Vietoris sequence we get the following cases:

1. $H_I^{h_1}(R) \cong H_{I_1}^{h_1}(R)$,
   $H_I^{h_2}(R) \cong H_{I_2}^{h_2}(R)$ and
   $H_I^{h_{12}-1}(R) \cong H_{I_1+I_2}^{h_{12}}(R)$.

2. $H_I^{h_1}(R) \cong H_{I_1}^{h_1}(R) \oplus H_{I_2}^{h_1}(R)$ and
   $H_I^{h_{12}-1}(R) \cong H_{I_1+I_2}^{h_{12}}(R)$.

3. $0 \longrightarrow H_{I_1}^{h_1}(R) \longrightarrow H_I^{h_1}(R) \longrightarrow H_{I_1+I_2}^{h_1+1}(R) \longrightarrow 0$ and
   $H_I^{h_2}(R) \cong H_{I_2}^{h_2}(R)$.

4. $0 \longrightarrow H_{I_1}^{h_1}(R) \oplus H_{I_2}^{h_1}(R) \longrightarrow H_I^{h_1}(R) \longrightarrow H_{I_1+I_2}^{h_1+1}(R) \longrightarrow 0$.

Applying the exact sequence of local cohomology to each case we obtain:

1. $H_{\mathfrak{m}}^p(H_I^{h_1}(R)) \cong H_{\mathfrak{m}}^p(H_{I_1}^{h_1}(R))$,
   $H_{\mathfrak{m}}^p(H_I^{h_2}(R)) \cong H_{\mathfrak{m}}^p(H_{I_2}^{h_2}(R))$ and
   $H_{\mathfrak{m}}^p(H_I^{h_{12}-1}(R)) \cong H_{\mathfrak{m}}^p(H_{I_1+I_2}^{h_{12}}(R))$.

2. $H_{\mathfrak{m}}^p(H_I^{h_1}(R)) \cong H_{\mathfrak{m}}^p(H_{I_1}^{h_1}(R)) \oplus H_{\mathfrak{m}}^p(H_{I_2}^{h_1}(R))$ and
   $H_{\mathfrak{m}}^p(H_I^{h_{12}-1}(R)) \cong H_{\mathfrak{m}}^p(H_{I_1+I_2}^{h_{12}}(R))$.

3. $0 \longrightarrow H_{\mathfrak{m}}^{n-h_1-1}(H_I^{h_1}(R)) \longrightarrow E_R(R/\mathfrak{m}) \longrightarrow E_R(R/\mathfrak{m}) \longrightarrow H_{\mathfrak{m}}^{n-h_1}(H_{I_1+I_2}^{h_1+1}(R)) \longrightarrow 0$
   and
   $H_{\mathfrak{m}}^p(H_I^{h_2}(R)) \cong H_{\mathfrak{m}}^p(H_{I_2}^{h_2}(R))$.

We want to see that $H_{\mathfrak{m}}^{n-h_1-1}(H_I^{h_1}(R))$ and $H_{\mathfrak{m}}^{n-h_1}(H_{I_1+I_2}^{h_1+1}(R))$ vanish. We have $CC(H_I^{h_1}(R)) = CC(H_{I_1}^{h_1}(R)) + CC(H_{I_1+I_2}^{h_1+1}(R))$. We can suppose $h_1 + 1 = n$ so:

$$CC(H_I^{n-1}(R)) = T^*_{\{x_1=\cdots=x_{n-1}=0\}}X + T^*_{\{x_1=\cdots=x_n=0\}}X.$$

Denote $M = H_I^{n-1}(R)$. Using the Čech complex

$$0 \longrightarrow M \longrightarrow \oplus M_{f_i} \longrightarrow \oplus M_{f_i f_j} \longrightarrow \cdots$$

and the result of [BMM94], see Section 2.1.2, we get

$$CC(M_{x_n}) = T^*_{\{x_1=\cdots=x_{n-1}=0\}}X + T^*_{\{x_1=\cdots=x_n=0\}}X,$$

and the characteristic cycle of the other localizations vanishes. Therefore the complex reduces to

$$0\longrightarrow M\longrightarrow M_{x_n}\longrightarrow 0,$$

and so $H^p_{\mathfrak{m}}(H^{h_1}_I(R))=0\ \forall p$.

4. $0\longrightarrow H^{n-h_1-1}_{\mathfrak{m}}(H^{h_1}_I(R)\longrightarrow E(R/\mathfrak{m})\longrightarrow E(R/\mathfrak{m})^2\longrightarrow H^{n-h_1}_{\mathfrak{m}}(H^{h_1+1}_{I_1+I_2}(R))\longrightarrow 0.$

We want to see that $H^{n-h_1-1}_{\mathfrak{m}}(H^{h_1}_I(R)$ vanishes and $CC(H^{n-h_1}_{\mathfrak{m}}(H^{h_1+1}_{I_1+I_2}(R)))=T^*_{\{x_1=\cdots=x_n=0\}}X$. As above, we can suppose $h_1+1=n$. Then we have

$$CC(H^{n-1}_I(R))=T^*_{\{x_1=\ldots=x_{n-1}=0\}}X+T^*_{\{x_1=\cdots=x_{n-2}=x_n=0\}}X+ \\ +T^*_{\{x_1=\cdots=x_n=0\}}X.$$

Denote $M=H^{n-1}_I(R)$. The Čech complex reduces to

$$0\longrightarrow M\longrightarrow M_{x_{n-1}}\oplus M_{x_n}\longrightarrow 0,$$

with:

$$CC(M_{x_{n-1}}\oplus M_{x_n})=T^*_{\{x_1=\cdots=x_{n-1}=0\}}X+T^*_{\{x_1=\ldots=x_{n-2}=x_n=0\}}X+ \\ +2T^*_{\{x_1=\ldots=x_n=0\}}X.$$

Therefore we get the desired result.

$m>2$.
Consider the Mayer-Vietoris sequence with
$U=I_1\cap\cdots\cap I_{m-1}$,
$V=I_m$,
$I=U\cap V=(I_1\cap\cdots\cap I_{m-1})\cap(I_m)$,
$U+V=I_1\cap\cdots\cap I_{m-1}+I_m$.
Then we have:

$$0\longrightarrow C_r\longrightarrow H^r_{U\cap V}(R)\longrightarrow A_{r+1}\longrightarrow 0.$$

Recall that $CC(C_r)\subseteq CC(H^r_U(R))+CC(H^r_V(R))$ and $CC(A_{r+1})\subseteq CC(H^{r+1}_{U+V}(R))$. By induction and Lemma 4.5 we have:

$$CC(H^r_U(R))=\sum_{I_i+I_m\in\Omega_{2,r}}CC(H^r_{I_i+I_m}(R))+\cdots+ \\ \sum_{I_1+\cdots+I_m\in\Omega_{m,r}}CC(H^{r+(m-3)}_{I_1+\cdots+I_{m-1}}(R)),$$

$$CC(H^r_V(R))=CC(H^r_{I_{m-1}+I_m}(R)),$$

$$CC(H^{r+1}_{U+V}(R))=\sum_{I_i+I_m\in\Omega^m_{2,r+1}}CC(H^{r+1}_{I_i+I_m}(R))+\cdots+ \\ \sum_{I_1+\cdots+I_m\in\Omega^m_{m,r+1}}CC(H^{r+(m-2)}_{I_1+\cdots+I_m}(R)).$$

To get the desired formula we only need to describe $CC(Z_p)$, so it is enough to prove the following

Claim:

$$CC(Z_p) = \sum T^*_{\{x_1=\cdots=x_n=0\}}X,$$

where the sum is taken over the cycles

$$T^*_{\{x_1=\cdots=x_n=0\}}X = CC(H^p_{\mathfrak{m}}(H^{r+j}_{I_{i_1}+\cdots+I_{i_j}+I_m}(R))) \in CC(H^p_{\mathfrak{m}}(A_{r+1})),$$

such that

$$T^*_{\{x_1=\cdots=x_n=0\}}X = CC(H^{p+1}_{\mathfrak{m}}(H^{r+j-1}_{I_{i_1}+\cdots+I_{i_j}}(R))) \in CC(H^{p+1}_{\mathfrak{m}}(C_r)).$$

The proof of the claim is as above.

□

Example:

Let $R = k[[x,y,z,t]]$. For $I = (xyz, xyt, xzt)$ we have already computed the sets $\Gamma_{j,r}$ of non paired and non almost paired sums of face ideals in the minimal primary decomposition of I with a given height. Therefore we can compute the Lyubeznik numbers of R/I.

We have $\Gamma_1 = \Gamma_1^1 = \{I_1\}$, where $\operatorname{ht} I_1 = 1$, so $CC(H^3_{\mathfrak{m}}(H^1_I(R))) = T^*_{\{x=y=z=t=0\}}X$. Therefore $\lambda_{3,3}(R/I) = 1$ and all the other Lyubeznik numbers vanish. The type is then:

$$\Lambda(R/I) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 1 \end{pmatrix}.$$

Recall that R/I is not Cohen-Macaulay.

Remark 4.6. Lyubeznik asked [Lyu93, Question 4.5] if $\lambda_{d,d}(R/I) = 1$ for all R/I where $d = \dim R/I$. Walther [Wal96] gave a negative answer when $d = 2$. We may give counterexamples for any dimension d as follows:

Consider $I = I_1 \cap \cdots \cap I_m$ such that $h_i > 1$ $\forall i$ and $\operatorname{ht}(I_{i_1} + \cdots + I_{i_s}) = \operatorname{ht} I_{i_1} + \cdots + \operatorname{ht} I_{i_s}$ $\forall s$. Then all the sum of face ideals are non paired and non almost paired, so $\lambda_{d,d}(R/I)$ is the number of face ideals in the minimal primary descomposition of height $n-d$.

Example:

Let $R = k[[x_1,x_2,x_3,x_4,x_5,x_6,x_7]]$ and $I = (x_1,x_2) \cap (x_3,x_4) \cap (x_5,x_6,x_7)$. Then the type of R/I is:

$$\Lambda(R/I) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 2 & 0 & 0 \\ & & & 0 & 1 & 0 \\ & & & & 1 & 0 \\ & & & & & 2 \end{pmatrix}.$$

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